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Special Solutions of the Fokker–Planck Transport Equation for Arbitrary Band Structures; Mobility and Diffusivity in a Uniform Field of Force

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A Fokker–Planck equation can be derived from a transition-type transport equation if the transition rates are nearly local in momentum space compared with the inhomogeneity length of the distribution. It is a second-order differential equation, whose coefficients depend on the band structure $E(\mathbf{k})$, the viscosity tensor $\eta(\mathbf{k})$, and the temperature T. Classical solutions of the Fokker–Planck equation deal with the parabolic band structure of free Brownian particles in a field of force. Mobility and diffusivity are then independent of the applied field. Here the explicit solution for the stationary state and the time-integrated conditional probability will be given in one dimension. This suffices to determine mobility and diffusivity. Assuming $\eta = 1$, these quantities become independent of the field and the band structure, if the latter is nonperiodic, though the distribution still depends on it. This property even holds in three dimensions for k-independent viscosity tensors. Field-dependent mobility and diffusivity are obtained for

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a k-dependent viscosity or $\eta = 1$ and periodic band structures. The latter is demonstrated for the case $E = -\cos k$, which is also related to the noise problem in Josephson junctions.

KEY WORDS: Nonequilibrium transport; Fokker–Planck equation; transport coefficients; noise-temperature conjecture; noisy Josephson junction.

1. INTRODUCTION

The Fokker–Planck equation has been studied extensively for classical systems in connection with Brownian motion and as a result of a linear Langevin equation (see Ref. 1). The fluctuation theory for nonlinear systems has been developed on the same basis by MacDonald[2] and Polder,[3] who introduced a velocity-dependent viscosity or friction coefficient in the Langevin equation. Inconsistencies coming out from a Langevin treatment of the nonlinear rate equations have been elucidated by van Kampen,[4] who claims that only a master-equation approach avoids the difficulties in deriving the noise spectrum of the nonlinear system. This is plausible since only the master equation defines all rate equations uniquely, but the velocity rate equation or its deterministic Langevin form with random force does not, in general, imply the velocity correlation function or the second velocity moment, quite apart from higher correlation functions and moments.

In the presence of a field of force the Fokker–Planck equation has been discussed by Kramers and others (see Ref. 5). The problem is quite similar to the transport equation for free electrons in an electric field, but instead of a discontinuous transition rate collision operator, a Fokker–Planck differential operator has to be used. We know that under certain conditions this can be justified (see Ref. 6). The discontinuous transitions can be approximated by a continuous stochastic process if the transition rates are sufficiently local compared to the inhomogeneity length of the distribution.

We should like to apply this concept to some typical collision operators. Consider the collisions of electrons by a deformation potential and acoustic phonons. A perturbation expansion with respect to the sound velocity allows reduction of the collision operator into a second-order differential term (continuous process) in energy (see Ref. 7), whereas the momentum change is still of transition type. If the transition rate is small for large momentum change, as, for example, in crystals with polar optical scattering (see Ref. 8), then a Fokker–Planck approximation in momentum can also be assumed. These approximations are justified only if the transition-rate multiplying factor is slowly varying in the range where the transition rate has a rather sharp maximum (method of steepest descent).

In this paper we give some general results for the transport problem as given by continuous stochastic processes. In Section 2 the transport equation with a Fokker–Planck collision term is derived from the usual transition-rate collision term. In Section 3 the steady-state solution and the mobility, and in Section 4 the diffusivity, are determined, respectively, for an arbitrary non-periodic band structure. The special case of a linear system is treated in Section 5. The example for a periodic band structure is given in Section 6, while Section 7 contains final remarks.

2. THE FOKKER–PLANCK EQUATION FOR CHARGED PARTICLES

The usual approach to transport theory of charged particles (see Ref. 9) includes the quantum mechanical properties of the band structure $E(\mathbf{k})$ resulting from a periodic lattice, and the transition rates $W_{\mathbf{k}\mathbf{k}'}$ for a transition of particles in the state $\mathbf{k}'$ to state $\mathbf{k}$. Since the pseudomomentum $\mathbf{k}$ increases in the presence of a field $\mathbf{F}$ $(d\mathbf{k}/dt = \mathbf{F})$, the change by flow in $\mathbf{k}$, $\mathbf{x}$ space will be compensated by collisions according to the transport equation

$$\frac{\partial f_{\mathbf{k}}}{\partial t} + \mathbf{F}\frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} + \mathbf{v}_{\mathbf{k}}\frac{\partial f_{\mathbf{k}}}{\partial \mathbf{x}} = \int d^3k'(W_{\mathbf{k}\mathbf{k}'}f_{\mathbf{k}'} - W_{\mathbf{k}'\mathbf{k}}f_{\mathbf{k}}) \tag{1}$$

where $\mathbf{v}_{\mathbf{k}} = \partial E_{\mathbf{k}}/\partial \mathbf{k}$ is the group velocity of the particles, $h_{\mathbf{k}} = C\exp(-E_{\mathbf{k}}/T)$ is the normalized Maxwell–Boltzmann distribution in equilibrium, and the transition rates fulfill the detailed balance $W_{\mathbf{k}\mathbf{k}'}h_{\mathbf{k}'} = W_{\mathbf{k}'\mathbf{k}}h_{\mathbf{k}}$, which ensures the equilibrium solution $f_{\mathbf{k}} = h_{\mathbf{k}}$ for $\mathbf{F} = 0$ $(\hbar = e = k_B = m = 1)$. As mentioned before, there are collision processes where transitions with small momentum changes $\mathbf{k} - \mathbf{k}'$ are predominant, i.e., the transition rates contribute only in a small range $|\mathbf{k} - \mathbf{k}'| \leqslant \Delta k$. If the distribution function is nearly constant over this range, the collision operator can be expanded in a power series of differential operators which has to be truncated at the second order for an equation with solutions of probability character (see Ref. 10). In order to have a Maxwell–Boltzmann distribution $h_{\mathbf{k}}$ in equilibrium, the complete equation for continuous collision processes becomes

$$\frac{\partial f}{\partial t} + \mathbf{F}\frac{\partial f}{\partial \mathbf{k}} + \mathbf{v}_{\mathbf{k}}\frac{\partial f}{\partial \mathbf{k}} = \frac{\partial}{\partial \mathbf{k}}\left[\eta_{\mathbf{k}}\left(v_{\mathbf{k}}f + T\frac{\partial f}{\partial \mathbf{k}}\right]\right) \tag{2}$$

where $\eta_{\mathbf{k}}$ is a momentum-dependent symmetric tensor, which is a friction coefficient or relaxation frequency, and gives the relaxation rates to the equilibrium distribution for different $\mathbf{k}$ values. According to Moyal,[10] $2\eta_{\alpha\beta}(\mathbf{k})$ is the second moment of $W_{\mathbf{k}\mathbf{k}'}$, and is even in the momentum, because the transition rate is invariant against $(\mathbf{k}, \mathbf{k}') \to (-\mathbf{k}, -\mathbf{k}')$.

For parabolic band structure and constant friction coefficient this equation has been discussed in the zero-field case and for a constant field. The result is a displaced Maxwellian distribution with displacement $u_\alpha = \bar{v}_\alpha = F_\alpha/\eta$ and a velocity–velocity correlation function $\overline{\Delta v(t)\Delta v(0)} = Te^{-\eta t}\delta_{\alpha\beta}$. Therefore mobility $\mu_{\alpha\beta}$ and diffusivity $D_{\alpha\beta}$,

$$\mu_{\alpha\beta} = d\bar{v}_\alpha/dF_\beta \tag{3}$$

$$D_{\alpha\beta} = \int_0^\infty \overline{\Delta v_\alpha(t)\Delta v_\beta(0)} \tag{4}$$

are independent of the field, in agreement with linear rate equations.

In the next section mobility and diffusivity will be determined formally for the general transport equation (2) in the case of nonperiodic band structures $E(\mathbf{k})$ in one dimension.

3. MOBILITY AND DIFFUSIVITY OF THE GENERAL FOKKER–PLANCK EQUATION IN ONE DIMENSION

The one-dimensional Fokker–Planck equation for a homogeneous system ($\partial/\partial x = 0$) becomes

$$\frac{\partial\phi}{\partial t} + F\frac{\partial\phi}{\partial k} = \frac{\partial}{\partial k}\left[\eta\left(v\phi + T\frac{\partial\phi}{\partial k}\right)\right] \tag{5}$$

and it has a nonegative, normalizable solution, since $\eta_k > 0$. In the stationary state ($\partial/\partial t = 0$) this equation reduces to

$$F\phi^s = \eta[v\phi^s + T(\partial\phi^s/\partial k)] \tag{6}$$

where the integration constant vanishes, because ϕ^s has to be normalizable in $-\infty < k < \infty$, and therefore $\phi^s \to 0$ for $|k| \to \infty$. Direct integration yields

$$\phi^s = C\exp[-(E_k - FB_k)/T], \qquad B_k = \int_0^k \frac{dk'}{\eta_{k'}} \tag{7}$$

and C follows from $\int dk\,\phi_s = 1$. The stationary solution exists, if $0 < \eta_k \leqslant \infty$, and is unique. It is stable,[2] since all initial distributions tend to ϕ^s for $t \to \infty$, i.e., deviations from ϕ^s decay to zero. According to Eq. (3), the mobility of the carriers is

$$\mu = du/dF, \qquad u = \bar{v}^s = \int dk\, v(k)\phi^s(k) \tag{8}$$

The explicit solution of the transient problem can only be given for simple cases, since a Laplace transform would give rise to an ordinary second-order differential equation. But in order to find D it is sufficient to

[2] Only in the range where $\lim_{k\to\infty} \eta v^2 = \infty$ does the existence of the stationary solution also imply its asymptotic stability (see appendix).

have the time integral of the conditional probability. Subtracting the stationary solution from (5) and integrating finally gives

$$\phi^s(k) - \delta(k - k_0) + F\frac{\partial \Delta}{\partial k} = \frac{\partial}{\partial k}\left[\eta\left(v\Delta + T\frac{\partial \Delta}{\partial k}\right)\right] \tag{9}$$

where

$$\Delta(k, k_0) = \int_0^\infty dt\,[\phi(kt|k_0) - \phi^s(k)] \tag{10}$$

By straightforward integration of (9), one gets

$$\begin{aligned}\Delta(k, k_0) = \{\exp[-(E(k) - FB(k))/T]\}\Big\{&B(k_0)\\ &+ \int_{-\infty}^{k} dk' \left[\int_{-\infty}^{k'} dk''\,(\phi^s(k'') - \delta(k'' - k_0))\right]\\ &\times \exp[(E(k') - FB(k'))/T]\Big\}\end{aligned} \tag{11}$$

where $B(k_0)$ follows from the condition $\int dk\,\Delta(k, k_0) = 0$, which can be verified by Eq. (10). Therefore $\Delta(k, k_0)$ is explicitly known for all band structures and friction coefficients.

According to Eq. (4), the diffusivity can be expressed by

$$D = \int dk \int dk_0\, v(k)v(k_0)\Delta(k, k_0)\phi^s(k_0) \tag{12}$$

and therefore the solution of (9) is sufficient in order to determine D.

The mobility $\mu(F)$ and the diffusivity $D(F)$ are therefore explicitly determined from the solutions of a Fokker–Planck equation in one dimension.

4. MOBILITY AND DIFFUSIVITY FOR THE CASE OF CONSTANT FRICTION IN THREE DIMENSIONS

Solution of the general Fokker–Planck equation in three dimensions is usually not possible in closed form. Even in the special case of constant friction coefficient the explicit solution cannot be found for arbitrary band structures. But it will now be proved that in this special case, mobility and diffusivity can very easily be derived from the lowest rate equations (moment equations) of the three-dimensional version of Eqs. (5) and (9).

The equation for the conservation of momentum is

$$(\partial\bar{\mathbf{k}}/\partial t) - \mathbf{F} = -\boldsymbol{\eta}\cdot\mathbf{v} \tag{13}$$

which in the stationary state provides drift velocity and mobility

$$\mathbf{u} = \bar{\mathbf{v}}^s = \boldsymbol{\eta}^{-1}\cdot\mathbf{F}, \qquad \boldsymbol{\mu} = \boldsymbol{\eta}^{-1} \tag{14}$$

a result which is well known for free electrons or parabolic bands.

The equivalent momentum equation from Eq. (9) is

$$\bar{\mathbf{k}}^s - \mathbf{k}_0 = -\eta \cdot \int d^3k\, \mathbf{v}\Delta \tag{15}$$

since the field term vanishes by $\int d^3k\Delta = 0$. The diffusivity is then, by Eqs. (12) and (15),

$$\begin{aligned} \mathsf{D} &= \int d^3k\, [\eta^{-1}\cdot(\mathbf{k} - \bar{\mathbf{k}}^s)]\mathbf{v}\phi^s(\mathbf{k}) \\ &= \int d^3k\, \eta^{-1}\cdot\mathbf{k}(\mathbf{v} - \bar{\mathbf{v}}^s)\phi^s(\mathbf{k}) \end{aligned} \tag{16}$$

which by use of Eqs. (6) and (14) becomes

$$\mathsf{D} = \int d^3k\, \eta^{-1}\cdot\mathbf{k}(-T\,\partial\phi^s/\partial\mathbf{k}) = \eta^{-1}T \tag{17}$$

This again is the well-known result for free electrons (see Ref. 5).

It is remarkable that mobility and diffusivity do not depend on the electric field nor on the band structure. This is a degeneracy in the sense that, though the distribution depends on F and $E(\mathbf{k})$, the lowest moment and correlation function coincide, respectively.

In all our considerations it was tacitly assumed that the band structure extends over the whole $\mathbf{k}$ space with $\lim_{|\mathbf{k}|\to\infty} E = \infty$, i.e., that $E(\mathbf{k})$ is non-periodic. Therefore the integration constant in Eq. (6) is zero if $|\mathbf{k}| \to \infty$ is inserted.

It will finally be shown that for periodic band structures the degeneracy mentioned is removed.

5. PERIODIC BAND STRUCTURE AND CONSTANT FRICTION

Explicit solutions of the Fokker–Planck equation and constant relaxation frequency ($\eta = 1$ is no restriction) can also be derived for periodic band structures in one dimension, i.e., $E_{k+p} = E_k$, where p is the period in the reciprocal lattice. Compared to Eq. (7), the stationary solution is

$$\phi^s = A\varphi_k \int_k^a (dk'/\varphi_{k'}), \qquad \varphi_k = \exp[-(E_k - Fk)/T] \tag{18}$$

since the integration constant, which vanishes in Eq. (6), is $A \neq 0$ here. The periodicity of ϕ^s is guaranteed if $a = \infty$. Summation over all periods gives

$$\phi^s = \frac{A}{1 - e^{-pF/T}}\,\varphi_k \int_k^{k+p} \frac{dk'}{\varphi_{k'}} \tag{19}$$

which can be decomposed into

$$\phi^s = \frac{1}{N}\varphi_k\left[\int_k^p \frac{dk'}{\psi_{k'}} + e^{-pF/T}\int_0^k \frac{dk'}{\varphi_{k'}}\right] \tag{20}$$

where N follows from normalization $\int_{(p)} dk\, \phi^s = 1$.

In the limit of $T = 0$ the first term is dominant and has a sharp maximum for $F < v_{\max}$ at $k = k_0(F)$, the smallest root of $v(k) = F$, if E_k has its absolute minimum in $k = 0$; for $F > v_{\max}$ the distribution can be found from the Fokker–Planck equation for $T = 0$,[3]

$$T = 0: \qquad \phi^s(k) = \begin{cases} \delta[k - k_0(F)], & |F| < v_{\max} \\ \dfrac{C}{1 - (v_k/F)}, & |F| > v_{\max} \end{cases} \tag{21}$$

In the limit $T \to \infty$ ($\varphi_k \to 1$) the particles are equally distributed and

$$T = \infty: \qquad \phi^s(k) = 1/p \tag{22}$$

The drift velocity obtained by simple integration of the Fokker–Planck equation gives

$$u(F, T) = F - Tp(1 - e^{-pF/T}) \times \left[\int_0^p dk\, \varphi_k \int_k^p \frac{dk'}{\varphi_{k'}} + e^{-pF/T}\int_0^p dk\, \varphi_k \int_0^k \frac{dk'}{\varphi_{k'}}\right]^{-1} \tag{23}$$

and

$$T = 0: \qquad u(F, 0) = \begin{cases} F, & |F| < v_{\max} \\ F - \dfrac{pF}{\int dk \dfrac{1}{1 - (v_k/F)}}, & |F| > v_{\max} \end{cases} \tag{24}$$

$$T = \infty: \quad u(F, \infty) = 0 \tag{25}$$

$$F \to \infty: \quad u(F, T) = (1/Fp)\int dk\, v^2 + O(1/F^3) \tag{26}$$

This determines the global behavior of the drift curves; details are given for $E = -\cos k$ in Fig. 1 for all temperatures. All curves have critical fields where the differential conductivity vanishes.

[3] The Fokker–Planck equation for $|F| < v_{\max}$ gives $\phi^s = \sum_n C_n \delta[k - k_n(F)]$, where n numbers the different solutions of $v_k = F$. What weights C_n have to be attributed to the solutions is not a priori evident. For $F = 0$ only the absolute energy minimum should contribute.

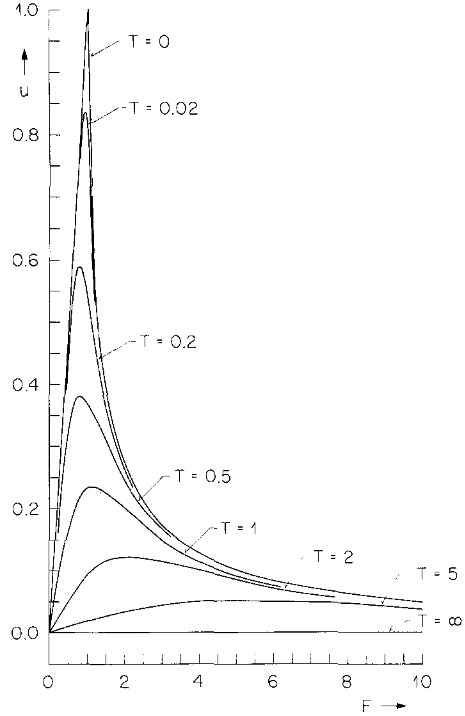

Fig. 1. Drift velocity u for the periodic band structure $E = -\cos k$ as a function of the applied field F with temperature T as a parameter.

There is a one-to-one correspondence between the drift in a periodic band of type $E = -\cos k$ and the noise problem of a Josephson oscillator with zero capacitance (see Ref. 11). The average voltage v equals

$$v(x, \gamma) = u(F, T) - F \tag{27}$$

where x is the input current, to be identified here with F, and $1/\gamma$ is the noise parameter, to be identified here with $T/2$. The average voltage input-current characteristic of a noisy Josephson junction is therefore essentially the average current input-voltage characteristic of a Fokker–Planck transport problem. An interference term in the Josephson noise problem corresponds to a k-dependent field and friction.[4]

The diffusion constant can be found in analogy to Section 3 for the nonperiodic case. Integration of (9) with $\eta = 1$ yields

$$T\frac{\partial \Delta}{\partial k} + v\Delta = -TB(k_0) + \int_{-p/2}^{k} dk' \, [\phi^s(k') - \delta_p(k - k_0)] \tag{28}$$

[4] The field term $F_x \, \partial f/\partial k$ in the transport equation has then to be written as $(\partial/\partial k) \times (F_{kx} f)$ in order to have local particle conservation.

where $\delta_p(k) = \sum_{n=-\infty}^{\infty} \delta(k - np)$, and $B(k_0)$ is determined by the normalization $\int_{(p)} dk \Delta = 0$. Since $\phi^s(k)$ and $\delta_p(k - k_0)$ are periodic in k, then $\Delta(k, k_0)$ also becomes a periodic function in k and k_0:

$$\Delta(k, k_0) = \frac{B(k_0)}{1 - e^{-pF/T}} \varphi_k \int_k^{k+p} \frac{dk'}{\varphi_{k'}} - \frac{\varphi_k}{T(1 - e^{-pF/T})} \times \int_k^{k+p} \frac{dk'}{\varphi_{k'}} \int_{-p/2}^{k'} dk'' \, [\phi^s(k'') - \delta_p(k'' - k_0)] \tag{29}$$

$$B(k_0) = \frac{1}{TN} \int_{(p)} dk \, \varphi_k \int_k^{k+p} \frac{dk'}{\varphi_{k'}} \int_{-p/2}^{k'} dk'' \, [\phi^s(k'') - \delta_p(k'' - k_0)] \tag{30}$$

Integrating (28) over the period p gives

$$\int_{(p)} dk \, v\Delta = k_0 - \bar{k}^s - pTB(k_0) \tag{31}$$

and the diffusivity D follows by means of (12).

This result has been applied again to $E = -\cos k$. The diffusion has been calculated numerically and Fig. 2 represents D and the noise temperature $\Theta = D/\mu$, which, according to Einstein's relation, has to coincide with T for $F = 0$. For $F \to \infty$ the diffusion constant decreases as

$$F \to \infty: \qquad D \sim T/2F^2 \tag{32}$$

which, because of $u \sim 1/2F$ [see Eq. (26)], gives formally $\Theta = -T$. It has to be mentioned that cases with negative mobility (i.e., $\Theta < 0$) are physically unstable with respect to small density fluctuations; therefore the negative Θ branch has been omitted in Fig. 2.

6. FINAL REMARKS

The Fokker–Planck equation has been introduced as a limit case of a transition-rate transport equation. This can be done if the transition rates are rather local compared to the inhomogeneity length of the distribution. This general Fokker–Planck equation has been studied in the special case of free particles or parabolic band structure in many classical papers.

Here we were especially interested in solutions for nonparabolic bands. Explicit solutions can be found in one dimension for the stationary distribution and for $\Delta(k, k_0)$ a time-integrated form of the conditional probability, which suffices for determining the diffusion constant.

In three dimensions the problem is analogous to the solution of the drifted diffusion equation, where drift and diffusion constant are space

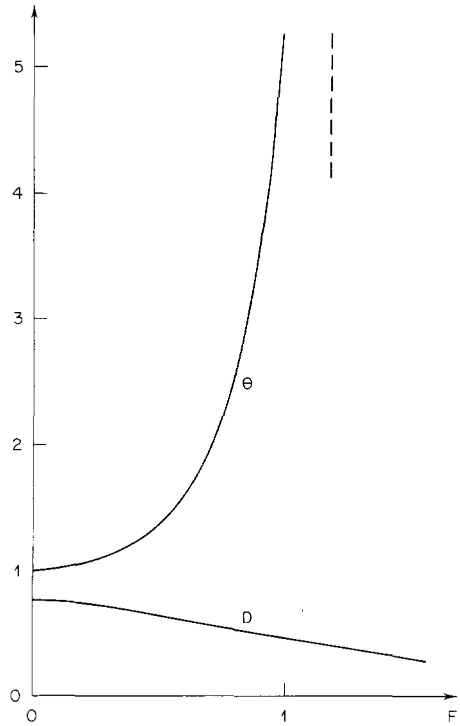

Fig. 2. Diffusion constant D and noise temperature $\Theta = D/(du/dF)$ as a function of the field F for temperature $T = 1$ ($k_B = 1$). At the critical field $F_c(T = 1) = 1.185$ the mobility vanishes.

dependent. Mobility and diffusivity cannot, in general, be found except in the case of a constant viscosity tensor $\eta_{\mathbf{k}} = \text{const}$.

The result for mobility and diffusivity then coincides with the same result of the well-known case of parabolic bands $E_{\mathbf{k}} = \mathbf{k}^2/2$, whereas the distribution and the conditional probability are quite different. This indicates a degeneracy, in the sense that averages (eigenvalues) are independent of certain properties (symmetries or parameters) though the distributions (wave functions) depend on it. Here mobility and diffusivity and therefore also the noise temperature Θ are independent of the field and the band structure, if the latter is nonperiodic.[5]

This degeneracy also has some connection to the noise conjecture (see Ref. 12), which claims that the noise temperature Θ is a minimum in equi-

[5] This does not hold for frequencies $\omega \neq 0$, except in the classical case of parabolic bands, where $\Theta(\omega, F, T) \equiv T$.

librium $F = 0$, where it coincides with T, the lattice temperature, whereas for $F > 0$ it should always be larger. For a parabolic band in the constant-τ approximation the result is $\Theta_{\parallel} = T + F^2\tau^2$ and $\Theta_{\perp} = T$, the latter degenerate with respect to the field.

A counterexample to the unqualified conjecture $\Theta_{\parallel}, \Theta_{\perp} \geqslant T$ was found by varying the band structure $E = |\mathbf{k}|^{\epsilon}$, since $\Theta_{\perp} = T +$ (positive constant) $\times\ (\epsilon - 2)F^2 + \cdots$ provides $\Theta_{\perp} < T$ for small fields and $\epsilon \lesssim 2$. This example led to a more precise formulation of the conjecture in three dimensions, the qualified conjecture, which so far has not been falsified.

Now the field degeneracy of the Fokker–Planck noise temperature $\Theta = T$ for parabolic bands could, hopefully, lead also to a genuine counterexample for the one-dimensional $\Theta \geqslant T$ conjecture if the energy is varied. Assuming $E_k = \frac{1}{2}k^2 + \lambda\epsilon_k$ (λ is a small expansion parameter), the noise temperature would be an expansion $\Theta = T + \lambda\Theta_1(F, T) + \lambda^2\Theta_2(F, T) + \ldots$, where the coefficients are functionals of ϵ_k. A counterexample would be relatively easily constructible if $\Theta_1 \not\equiv 0$. But explicit perturbation theory not only gives $\Theta_1 = 0$ but also $\Theta_2 = \Theta_3 = \cdots = 0$, for which a much simpler proof was found later, which has been presented in Section 4.

APPENDIX

According to Eq. (7) the stationary solution ϕ^s exists for all fields if (1) $\eta_k > 0$ in $-\infty < k < \infty$ (eventually $\lim_{k\to\infty} \eta_k = 0$), and if (2) $\lim_{k\to\infty} E_k/B_k = \infty$. If $\lim_{k\to\infty} E_k/B_k = F_R$ is finite, the stationary solution exists only for $|F| < F_R$, a runaway field.

The solution of the time-dependent equation (5) can be reduced by $\phi \propto e^{-\lambda t}$ to the eigenvalue problem

$$T\frac{\partial}{\partial k}\left(\eta\frac{\partial\phi}{\partial k}\right) + \frac{\partial}{\partial k}(\eta v - F)\phi + \lambda\phi = 0 \tag{A.1}$$

which by substituting $\psi = \phi/\phi^s$ is equivalent to the (self-adjoint) Sturm–Liouville equation

$$(T\eta\phi^s\psi')' + \lambda\phi^s\psi = 0 \tag{A.2}$$

All eigenvalues λ must be nonnegative, since $\int dk\, \psi\{(\text{A.2})\}$ gives immediately

$$\lambda = T\int dk\, \eta\phi^s\psi'^2 \Big/ \int dk\, \phi^s\psi^2 \geqslant 0 \tag{A.3}$$

where the value $\lambda = 0$ belongs to the stationary solution $\phi = \phi^s$ or $\psi = 1$. If the lowest eigenvalue were well separated from the next, only the $\lambda = 0$ term would survive for $t \to \infty$ if the solution $\phi(k, t)$ were expanded with respect to eigenfunctions.

A Schrödinger representation of the eigenvalue problems gives more insight into the nature of the eigenvalue spectrum. The substitutions

$$\phi = \eta^{-1/4}(k)[\phi^s(k)]^{1/2}\varphi(x), \qquad x(k) = \int_0^k \frac{dk'}{\eta^{1/2}(k')} \tag{A.4}$$

transform (A.1) into

$$T\, d^2\varphi/dx^2 + (\lambda - V(x))\varphi = 0 \tag{A.5}$$

with the potential

$$V(x) = \frac{T}{4}\left[\frac{d^2\eta}{dk^2} - \frac{(d\eta/dk)^2}{\eta}\right] + \frac{(\eta v - F)^2}{4T\eta} - \frac{1}{2}\frac{d}{dk}(\eta v) \tag{A.6}$$

If the x range is infinite and $V(x) \to +\infty$ for $|x| \to \infty$ or if the range is finite $-x_1 < x < x_1$, the eigenvalues $\{\lambda_n\}$ are all discrete and therefore $\lambda_0 = 0$ (simple, since the stationary solution is unique) and $\lambda_n \geqslant \lambda_1 > 0$ are well separated.

Assuming asymptotic forms $E \sim |k|^\epsilon$ with $\epsilon > 0$ and $\eta \sim |k|^\alpha$, the x range becomes finite if $\alpha > 2$, $\epsilon > 0$. For infinite x range, $\alpha \leqslant 2$, and $\eta v^2 \sim k^{\alpha+2\epsilon-2}$ is the dominant term for $x \to \infty$ in the potential. The stationary solution exists if $E_k/B_k \sim k^{\epsilon+\alpha-1} \to \infty$, i.e., for $\epsilon > 0$ and $\epsilon + \alpha > 1$, whereas asymptotic stability follows for $\epsilon > 0$ and $\epsilon + \frac{1}{2}\alpha > 1$, i.e., if $\lim_{k\to\infty} \eta v^2 = \infty$. In the range $\epsilon > 0$, $\epsilon + \alpha > 1$, $\epsilon + \frac{1}{2}\alpha < 1$, where $\lim_{k\to\infty} \eta v^2 = 0$, the eigenvalue spectrum is continuous starting from $\lambda = 0$ and stability needs a more detailed discussion.

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